Midterm Exam I - Review
MATH 125 - Spring 2022

Sections 2.1-2.5, 2.7, 2.8, 3.1-3.3, 3.7, 3.8

Midterm Exam 1: Tuesday 3/1, 5:50-7:50 pm in Strong 330 and Wescoe 3140

The following is a list of important concepts that will be tested on Midterm Exam 1. This is not a complete list of the material that you should know for the course, but it is a good indication of what will be emphasized on the exam. A thorough understanding of all of the following concepts will help you perform well on the exam. Some places to find problems on these topics are the following: in the book, in the slides, in the homework, on quizzes, and WebAssign.

- **Average and Instantaneous Rates of Change**

Total change	Average change	Instantaneous change
Total change of $f(x)$ on $[a, a+h]$	Average rate of change of $f(x)$ on $[a, a+h]$	Instantaneous rate of change of $f(x)$ at a
Change in y	Difference Quotient	Derivative
$f(a+h) - f(a)$	$\dfrac{f(a+h) - f(a)}{h}$	$\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$
Change in Position	Average Velocity	(Instantaneous) Velocity
	Slope of secant line	Slope of tangent line

1. A ball is thrown into the air. The height (in feet) of the ball at a time t (in seconds) is given by

$$p(t) = -16t^2 + 10t + 7$$

(a) Find the average velocity of the ball between times $t = 1$ and $t = 2$.

(b) Find the average velocity of the ball between times $t = 1$ and $t = h$.

(c) Find the velocity of the ball at time $t = 1$.

(a) $\dfrac{p(2) - p(1)}{2 - 1} = \dfrac{-37 - 1}{1} = -38\, ft/s$ (b) $\dfrac{p(h) - p(1)}{h - 1} = \dfrac{-16h^2 + 10h + 6}{h - 1}\, ft/s$

(c) $v(1) = s'(1) = \lim\limits_{x\to 1} \dfrac{p(x) - p(1)}{h - 1} = \lim\limits_{x\to 1} \dfrac{-16x^2 + 10x + 6}{x - 1}$

$= \lim\limits_{x\to 1} \dfrac{-2(x-1)(8x+3)}{x - 1} = -2(8 + 3) = -22\, ft/s$

Alternatively, by derivative rules $v(t) = -32t + 10$ and $v(1) = 22 ft/sec$

2. Find the secant line to the function $f(x) = x^3 - 1$ on the interval $[-2, 3]$.

Two points $(-2, f(-2)) = (-2, -9)$ and $(3, f(3)) = (3, 26)$.

Slope: $\dfrac{f(3) - f(-2)}{3 - (-2)} = \dfrac{26 - -(-9)}{5} = \dfrac{35}{5} = 7$

Secant line: $y - (-9) = 7(x - (-2)) \implies y = 7x + 5$

- **Limits of Functions**
  - Understand the definitions of limit; this includes limits from the left, limits from the right, and infinite limits.
  - Find vertical asymptotes for functions.
  - Sketch a function satisfying a list of limit properties.
  - Compute limits of various functions, including limits from the left and right both graphically and algebraically. Use the limit laws and limit calculation techniques: direct substitution, simplification, conjugation, and the Squeeze Theorem.
  - Compute limits of piecewise defined functions using limits from the left and right.
  - Compute limits at $\pm\infty$. Keep in mind the idea of multiplying through by 1 over the dominant term in the denominator. Find horizontal asymptotes using limits as $x \to \infty$ and $x \to -\infty$.

1. Compute the following limits if they exist. If the limit is infinite, specify if it is ∞, $-\infty$, or does not exist.
2. There are 7 indeterminate forms:

$$\frac{0}{0} \qquad \pm\frac{\infty}{\infty} \qquad \infty - \infty \qquad \pm 0 \cdot \infty \qquad 1^{\infty} \qquad 0^0 \qquad \infty^0$$

(a) $\displaystyle\lim_{x\to 2} \frac{x^2+3x-10}{x^2-3x+2}$

Form $\frac{0}{0}$ so simplify;

$$= \lim_{x\to 2} \frac{(x+5)\cancel{(x-2)}}{\cancel{(x-2)}(x-1)} = \lim_{x\to 2} \frac{x+5}{x-1} = 7$$

(b) $\displaystyle\lim_{x\to 5} \frac{x^2-6x+5}{x+5}$

Direct Substitute:

$$= \frac{0}{10} = 0$$

(c) $\displaystyle\lim_{x\to -3} \frac{\sqrt{x^2+7}-4}{x+3}$

Indeterminate form: $\frac{0}{0}$

Multiply both numerator and denominator by conjugate:

$$\lim_{x\to -3} \frac{\sqrt{x^2+7}-4}{x+3} \times \frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4}$$

Simplify:

$$= \lim_{x\to -3} \frac{x^2-9}{(x+3)\left(\sqrt{x^2+7}+4\right)}$$

$$= \lim_{x\to -3} \frac{\cancelto{1}{(x+3)}(x-3)}{\cancelto{1}{(x+3)}\left(\sqrt{x^2+7}+4\right)}$$

$$= \lim_{x\to -3} \frac{x-3}{\sqrt{x^2+7}+4} \quad \Longrightarrow$$

$$\text{Answer} = \lim_{x\to -3} \frac{x-3}{\sqrt{x^2+7}+4} = \boxed{-\frac{3}{4}}$$

$f(x) = \sqrt{x^2+7}$ and $a = -3$.

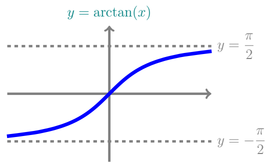
(d) $\displaystyle\lim_{x\to 3^-} \arctan\left(\frac{1}{3-x}\right)$

Note: $\lim_{x\to 3^-} \left(\frac{1}{3-x}\right) = \infty$

Now

$$\lim_{x\to 3^-} \arctan\left(\frac{1}{3-x}\right)$$

$$= \lim_{u\to\infty} \arctan(u) = \frac{\pi}{2}$$



(e) $\displaystyle\lim_{x\to -3} \frac{\frac{1}{3} - \frac{1}{x}}{x+3}$

This is of the form $\frac{2/3}{0}$ which is ∞.

(f) $\displaystyle\lim_{x\to -3} \frac{\frac{1}{3} + \frac{1}{x}}{x+3}$

We use Simplification on this limit of

the form $\dfrac{0}{0}$:

$$\underset{\text{Form}\frac{0}{0}}{=} \lim_{x\to -3} \frac{\overbrace{\dfrac{x+3}{3x}}^{\text{Common Denominator}}}{x+3}$$

$$\underset{\text{Simplify}}{=} \lim_{x\to 3} \frac{3+x}{3x} \times \frac{1}{3+x}$$

$$= \lim_{x\to -3} \frac{\cancel{(x+3)}^{1}}{3x\cancel{(x+3)}^{1}}$$

$$= \lim_{x\to -3} \frac{1}{3x} \underset{\text{Substitute}}{=} \boxed{\frac{-1}{9}}$$

(g) $\displaystyle\lim_{x\to-\infty} \frac{2e^x + e^{-x}}{3e^x + 7e^{-x}}$

Remember: $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} e^{-x} = 0$:

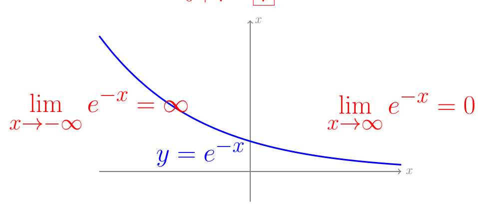
Then $\underbrace{\lim_{x\to-\infty} e^{-x} = \infty \text{ and } \lim_{x\to-\infty} e^{x} = 0}$

Multiply numerator and denominator by $\frac{1}{e^{-x}}$

$$\lim_{x\to-\infty} \frac{2e^x + e^{-x}}{3e^x + 7e^{-x}} =$$

$$\lim_{x\to-\infty} \frac{2e^{2x} + 1}{3e^{2x} + 7} =$$

$$\frac{0+1}{0+7} = \boxed{\frac{1}{7}}$$



(h) $\displaystyle\lim_{x\to 0} x^3 \sin\left(\frac{1}{x}\right)$

For all $x \neq 0$,

$-|x^3| \leq x^3 \sin\left(\dfrac{1}{x}\right) \leq |x^3|$.

Now,

$$\underbrace{-|x^3|}_{=f(x)} \leq \underbrace{x^3 \sin\left(\frac{1}{x}\right)}_{=g(x))} \leq \underbrace{|x^3|}_{=h(x)}$$

$$\lim_{x\to 0} f(x) = 0 \text{ and } \lim_{x\to 0} h(x) = 0 \implies$$

By, Squeeze Theorem, $\lim_{x\to 0} g(x) = 0$.

(i) $\displaystyle\lim_{x\to -2} \frac{2-|x|}{2+x}$

$$\frac{2-|x|}{2+x} = \begin{cases} \dfrac{2-(-x)}{2+x} & x \leq 0 \\[2ex] \dfrac{2-x}{2+x} & x > 0 \end{cases}$$

$\implies$

$$\frac{2-|x|}{2+x} = \begin{cases} \dfrac{2+x}{2+x} & x \leq 0 \\[2ex] \dfrac{2-x}{2+x} & x > 0 \end{cases}$$

$\implies$

$$\frac{2-|x|}{2+x} = \begin{cases} 1 & \underbrace{x \neq -2 \text{ and } x \leq 0}_{\text{Points around} x=-2 \text{are in here.}} \\[2ex] \dfrac{2-x}{2+x} & x > 0 \end{cases}$$

$\lim_{x\to -2} f(x)$ is a limit at the cut off point :

Find left and right limits.

$\lim_{x\to -2^-} f(x) = 1$ and $\lim_{x\to -2^+} f(x) = 1$

$\implies \boxed{\lim_{x\to -2} f(x) = 1}$

3. Find the horizontal asymptotes, if any exist, for the following function:

$$f(x) = \sqrt{x^2 + 2x - 2} - \sqrt{x^2 - 2}$$

$\lim_{x\to\infty} f(x)$ is of the **indeterminate form**: $\boxed{\infty - \infty}$

Multiply by conjugate to change the form:

$$\lim_{x\to\pm\infty} \sqrt{x^2+2x-2} - \sqrt{x^2-2} =$$

$$\lim_{x\to\pm\infty} \frac{(\sqrt{x^2+2x-2} - \sqrt{x^2-2})(\sqrt{x^2+2x-2} + \sqrt{x^2-2})}{\sqrt{x^2+2x-2} + \sqrt{x^2-2}} =$$

$$\lim_{x\to\pm\infty} \frac{x^2+2x-2-(x^2-2)}{\sqrt{x^2+2x-2} + \sqrt{x^2-2}} =$$

$$\lim_{x\to\pm\infty} \frac{2x}{\sqrt{x^2+2x-2} + \sqrt{x^2-2}}$$

This one now is of the indeterminate form $\frac{\infty}{\infty}$: Multiply by the reciprocal of the highest power of denominator $\frac{1}{x}$.

As $x \to \pm\infty$: $\sqrt{x^2} = |x| = \pm x$.

$$= \lim_{x\to\pm\infty} \frac{2x}{|x|\left(\sqrt{1+\frac{2}{x}-\frac{2}{x^2}}\right) + |x|\sqrt{1-\frac{2}{x^2}}} =$$

Multiply numerator and denominator.

$$\lim_{x\to\pm\infty} \frac{2x}{\pm x\left(\sqrt{1+\frac{2}{x}-\frac{2}{x^2}}\right) \pm x\sqrt{1-\frac{2}{x^2}}} =$$

$\lim_{x\to\pm\infty} \frac{1}{x} = 0$

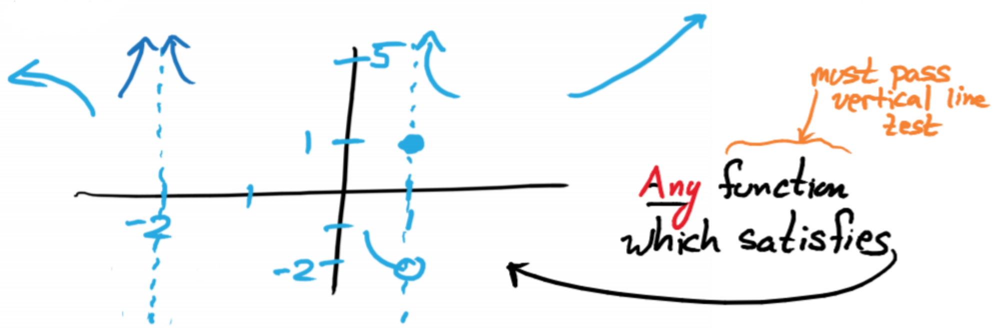
$$\lim_{x\to-\infty} \frac{2}{\pm 1 \pm 1} = \boxed{\pm 1}$$

A horizontal asymptote is $\boxed{y=1}$ as $x \to \infty$ and a horizontal asymptote is $\boxed{y=-1}$ as $x \to -\infty$

3. Sketch the graph of a function f that satisfies the following properties:

$$f(1) = 1 \qquad \lim_{x\to 1^+} f(x) = \infty \qquad \lim_{x\to 1^-} f(x) = -2$$

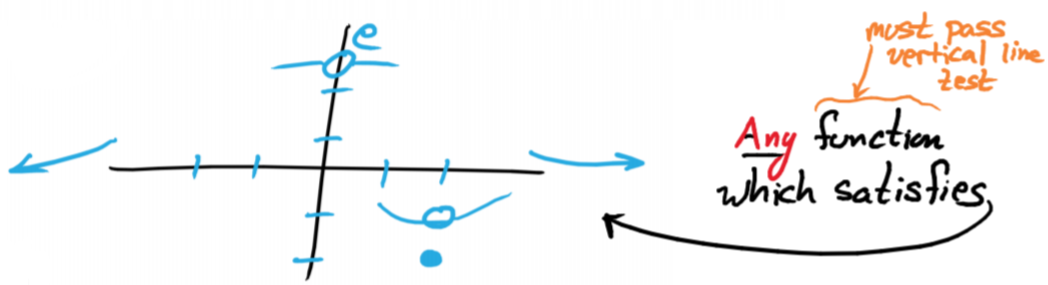
$$\lim_{x\to-\infty} f(x) = 5 \qquad \lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to -2} f(x) = \infty$$



4. Sketch the graph of a function f that satisfies the following properties:

$$f(2) = -2 \qquad \lim_{x\to 2} f(x) = -1 \qquad f(0) \text{ DNE}$$

$$\lim_{x\to 0} f(x) = e \qquad \lim_{x\to\infty} f(x) = 0 \qquad \lim_{x\to-\infty} f(x) = 0$$



- **Continuity and the Intermediate Value Theorem**
  - **Definition:** A function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.
  - Identify different types of discontinuities graphically and algebraically:

Removable Discontinuity (hole)	Jump Discontinuity	Infinite Discontinuity
$\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} f(x) \neq f(a)$	$\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$	$\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

  - Use continuity to evaluate limits. If $f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$
  - Use the Intermediate Value Theorem to show the existence of solutions to equations involving continuous functions.

1. Give all the x-values where the function has discontinuities and identify the type of discontinuity.

$$f(x) = \begin{cases} 3x^2 - 2x - 3 & x \leq 0 \\ x - 3 & 0 < x < 4 \\ x^2 - 3x - 3 & x \geq 4 \end{cases} \qquad g(x) = \begin{cases} \cos(x) & x < 0 \\ 0 & x = 0 \\ x^2 - 1 & 0 < x < 1 \\ \dfrac{1}{x-2} & x \geq 1 \end{cases}$$

$$f(x) = \begin{cases} 3x^2 - 2x - 3 & x \leq 0 \\ x - 3 & 0 < x < 4 \\ x^2 - 3x - 3 & x \geq 4 \end{cases}$$

All pieces of $f(x)$ are continuous inside their domain but the end points have to be checked.

Checking Continuity at $x = 0$:
$\lim_{x\to 0} 3x^2 - 2x - 3 = -3 = \lim_{x\to 0} x - 3$ so f is continuous at $x = 0$.

Checking Continuity at $x = 4$:
$\lim_{x\to 4} x - 3 = 1 = \lim_{x\to 4} x^2 - 3x - 3$ so f is continuous at $x = 4$.

So f is continuous everywhere.

$$g(x) = \begin{cases} \cos(x) & x < 0 \\ 0 & x = 0 \\ x^2 - 1 & 0 < x < 1 \\ \dfrac{1}{x-2} & x \geq 1 \end{cases}$$

All pieces of $f(x)$ are continuous inside their domain but at $x = 2$ and the end points have to be checked:

Checking Continuity at $x = 0$:
$\lim_{x\to 0} \cos(x) = 1 \neq \lim_{x\to 0} x^2 - 1 = -1 \neq f(0) = 0$. So f is **NOT** continuous at $x = 0$. A jump discontinuity at $x = 0$.

Checking Continuity at $x = 1$:
$\lim_{x\to 1} x^2 - 1 = 0 \neq -1 \lim_{x\to 1} \dfrac{1}{x-2}$ so f is **NOT** continuous at $x = 4$. Jump discontinuity at $x = 1$.

Infinite discontinuous at $x = 2$.

2. In 1987 it cost 22 cents to mail a letter first class inside the US and in 1990 it cost 25 cents to mail the same letter. Can we conclude that the cost to mail a letter was 23 cents at some point in time?

The cost function is not continuous so we can not use IVT.

3. If a child's temperature rose from $98.6°F$ to $101.3°F$, was there an instant that the child's temperature was $100°F$? (Compare this to the previous problem — what is the difference?)

A child's temperature is continuous and $98.6°F < 100°F < 101.3°F$ and by mean value theorem, there is a t during that time period that their temperature was $100°F$.

4. Using the Intermediate Value Theorem and bisection, approximate the roots of the function $f(x) = x - x^3 + 1$ accurate to one decimal point.

$f(2) < 0$ and $f(1) > 0$. f is continuous on $[1, 2]$. Therefore, we can use the bisection method on that interval.

Interval	Midpoint	y=value for midpoint	Interval Containing the root	Length of the interval
$[1^+, 2^-]$	1.5	$-0.875 < 0$	$[1, 1.5]$	$\frac{1}{2}$
$[1^+, 1.5^-]$	1.25	$2.01515 > 0$	$[1.25, 1.5]$	$\frac{1}{4}$
$[1.25^+, 1.5^-]$	1.375	$-0.22461 < 0$	$[1.25, 1.375]$	$\frac{1}{8}$
$[1.25^+, 1.375^-]$	1.3125	$0.05151 > 0$	$[1.3125, 1.375]$ The interval	$\frac{1}{16}$

Accurate to one decimal place is 1.3

5. Does the Intermediate Value Theorem guarantee that $g(x) = \dfrac{1}{x}$ has a root on the interval $[-1, 1]$?

No, $g(x)$ is not continuous on $[-1, 1]$.

6. Find the values a and b which make the following function continuous everywhere:

$$f(x) = \begin{cases} x^2 - 2x + a & \text{if } x < -2 \\ b & \text{if } x = -2 \\ \dfrac{1}{x+4} & \text{if } x > -2 \end{cases}$$

$x^2 - 2x + a$ is continuous everywhere. $\dfrac{1}{x+4}$ is continuous on all values $x > -2$. So what remains is to check the end points.

Conditions for continuity at $x = -2$:

$$\lim_{x\to -2^-} x^2 - 2x + a = 8 + a = f(-2) = \lim_{x\to -2^+} \frac{1}{x+4} = \frac{1}{2}$$

So $b = \frac{1}{2}$ and $a = -\frac{15}{2}$

- **Definition of the Derivative**
  - Compute derivatives of common functions (polynomials, rational functions, and square roots) using the limit definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

  - Know the typical ways a function can fail to be differentiable: corners, cusps, vertical tangents, and discontinuities.
  - Find the equations of tangent lines to curves.

1. Use the limit definition of the derivative in the following problems.

   (a) Compute $f'(1)$ for the function $f(x) = x^2 - x + 2$.

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1} \frac{x^2 - x + 2 - 2}{x-1} \\ &= \lim_{x\to 1} \frac{x(x-1)}{x-1} = \lim_{x\to 1} x = \boxed{1} \end{aligned}$$

   (b) Compute $f'(x)$ for the function $f(x) = \frac{1}{x}$. What is the domain of $f'(x)$?

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{x-1} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{x - (x+h)}{x(x+h)h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \boxed{-\frac{1}{x^2}} \end{aligned}$$

Domain of $f'(x)$

$(-\infty, 0) \cup (0, \infty)$

2. Let $P(t)$ be the population of China (in billions), where t is the number of years since 1965. What does it mean when $P'(30) = 0.15$?

The instantaneous rate of change in population in 1995 was 0.15 billion additional people per year. (The figure 0.15 may not be accurate.)

3. The total cost of producing x feet of rope is $C(x)$ dollars.

   (a) What are the units of $C'(x)$?

$$C'(a) = \lim_{x\to a} \frac{C(x) - C(a)}{x-a} \quad \boxed{\frac{\text{dollars}}{\text{ft}}}$$

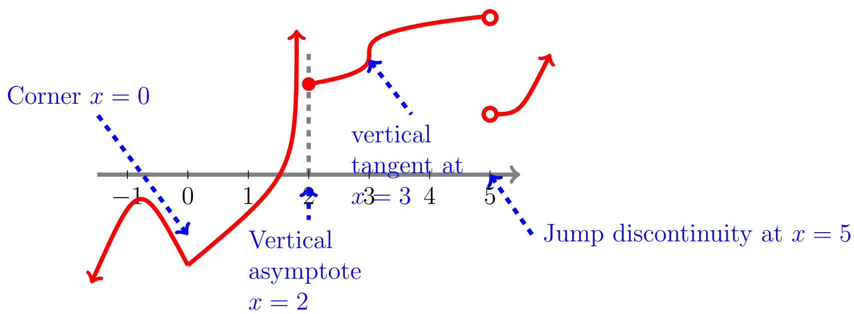
(b) What is the practical meaning of $C'(100) = 1.4$?

$C'(100) = 1.4$ The instantaneous change in cost in producing rope after 100 ft is \$1.4 per feet.

(c) Suppose $C(100) = 800$ and $C'(100) = 1.4$. Estimate $C(110)$.

$C(10) \approx C(100) + 10C'(100) = 800 + 10(1.4) = \$814$

4. Identify the values where the function graphed below is **not** differentiable. Classify the reason why f is not differentiable at each value.



- **Current List of Derivative Rules**

$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$	$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x)+f(x)g'(x)$ (product rule)
$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$ (quotient rule)
$\frac{d}{dx}(c\,f(x)) = cf'(x)$	$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (chain rule)
$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = n\,x^{n-1}$ (power rule)
$\frac{d}{dx}(a^x) = a^x \ln(a)$	$\frac{d}{dx}(e^x) = e^x$

1. Use the table to compute the following derivatives. Be aware that you may not have enough information to evaluate a derivative.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	1	4	0	-1
4	3	3	-1	0

(a) If $h(x) = f(x^2)$, find $h'(2)$.

$h'(x) = 2xf'(x^2)$ $\qquad$ $h'(2) = 2(2)f'(4) = 4(3) = 12$

(b) If $h(x) = x^2 f(x)$, find $h'(4)$.

$h'(x) = 2xf(x) + x^2 f'(x)$ $\qquad$ $h'(4) = 8(3) + 16(3) = 72$

(c) If $h(x) = f(x)g(x)$, find $h'(2)$.

$h'(x) = f'(x)g(x) = f(x)g'(x)$ $\qquad$ $h'(2) = (4)(0) + (1)(-1) = -1$

(d) If $h(x) = \dfrac{f(x)}{g(x)}$, find $h'(4)$.

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \qquad h'(4) = \frac{(-1)(2) - (0)(3)}{(-1)^2} = -3$$

(e) If $h(x) = \dfrac{f(x) - x}{g(x-1)}$, find $h'(2)$.

$$h'(x) = \frac{g(x-1)(f'(x)-1) - g'(x-1)(f(x)-x)}{(g(x-1))^2} \qquad h'(2) = \frac{g(2-1)(f'(2)-1) - g'(2-1)(f(2)-2)}{(g(2-1))^2}$$

Can not be done since $g'(1)$ and $g(1)$ are not given.

2. Find an equation for the tangent line to the curve $y = \sqrt{e^x + 8}$ at $x = 0$.

$f(x) = \sqrt{e^x+8}$

$f'(x) = \dfrac{e^x}{2\sqrt{e^x+8}}$

Point $f(0) = 3$

$f'(0) = 1/6$ **Slope**

$y = 1/6\, x + 3$ **Tangent Line**

3. Differentiate the following functions and simplify the results.

(a) $f(x) = x^3 - 2x^\pi - \pi^x - x^{-1}$

$f'(x) = 3x^2 - 2\pi x^{\pi-1} - \pi^x \ln(\pi) + x^{-2}$

Power Rule

(b) $g(x) = \dfrac{x^3 - x + 2}{x+1}$ **Quotient Rule**

$g'(x) = \dfrac{(x+1)(3x^2-1) - (x^3-x+2)(1)}{(x+1)^2}$

$= \dfrac{2x^3+3x^2-3}{(x+1)^2}$

(c) $h(x) = \sqrt{x}(x^3 - 1)$ **Product Rule**

$h'(x) = \dfrac{1}{2\sqrt{x}}(x^3-1) + \sqrt{x}(3x^2) = \dfrac{7x^3-1}{2\sqrt{x}}$

(d) $r(x) = e^{2x^2-6x+1}$ **Chain Rule**

$r'(x) = (4x-6)e^{2x^2-6x+1}$

(e) $s(t) = \frac{t}{\sqrt{t^2+1}}$ **Quotient then Chain Rules**

$s'(t) = \dfrac{\sqrt{t^2+1}\,(1) - t\,\frac{2t}{2\sqrt{t^2+1}}}{t^2+1}$

$= \dfrac{1}{(t^2+1)^{3/2}}$

(f) $v(t) = (t^4 - 1)^3(t^3 + 1)^{-2}$

$v'(t) = 3(t^4-1)^2(4t^3)(t^3+1)^{-2} + (t^4-1)^3(-2)(t^3+1)^{-3}(3t^2)$

$= (t^4-1)^2(t^3+1)^{-3}\left[12t^3(t^3+1) - 6t^2(t^4-1)\right]$

$= \dfrac{6t^2(t^4-1)^2(t^4+2t+1)}{(t^3+1)^3}$

Product then 2 Chain Rules

(g) $S(z) = \sqrt{\dfrac{z-7}{z+7}}$ **Chain then Quotient Rules**

$S'(z) = \frac{1}{2}\left(\frac{z-7}{z+7}\right)^{-1/2} \dfrac{(z+7)\frac{d}{dz}(z-7) - (z-7)\frac{d}{dx}(z+7)}{(z+7)^2}$

$= \frac{1}{2}\left(\frac{z+7}{z-7}\right)^{+1/2} \dfrac{14}{(z+7)^2} = \dfrac{7}{\sqrt{z-7}\,\sqrt{z+7}^{\,3}}$

(h) $V(x) = \sqrt{x + \sqrt{2x + \sqrt{3x}}}$ $V'(x) = \frac{1}{2}\left(x+\sqrt{2x+\sqrt{3x}}\right)^{-1/2}\left(1+\frac{1}{2}\left(2x+\sqrt{3x}\right)^{-1/2}\left(2+\frac{1}{2}(3x)^{-1/2}\cdot 3\right)\right)$

derivative derivative derivative

4. Find a polynomial P of degree 2 such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

find a, b, c

$P(x) = ax^2 + bx + c$
$P'(x) = 2ax + b$
$P''(x) = 2a$

$P''(2) = 2 \Rightarrow a = 1$
$P'(2) = 3 \Rightarrow 3 = 4 + b$
$\Rightarrow b = -1$
$P(2) = 5 \Rightarrow 5 = 4 - 2 + c$
$\Rightarrow c = 3$

$y = x^2 - x + 3$

5. The functions

$$y = x^2 + ax + b \qquad y = cx - x^2$$

share a tangent line at the point $(1, 0)$. Find a, b, and c.

If the functions share a tangent line at $(1,0)$
then (i) $(1,0)$ lies on the curve of $y = x^2 + ax + b$
$\Rightarrow 0 = 1 + a + b$

(ii) $(1,0)$ lies on the curve of $y = cx - x^2$
$\Rightarrow 0 = c - 1 \Rightarrow c = 1$

(iii) The derivatives at $x=1$ both are the slope of the tangent lines:
$\Rightarrow \frac{d}{dx}(x^2 + ax + b)\big|_{x=1} = \frac{d}{dx}(cx - x^2)\big|_{x=1}$
$\Rightarrow 2x + a\big|_{x=1} = c - 2x\big|_{x=1}$
$\Rightarrow 2 + a = c - 2 \Rightarrow a = c - 4 = -3$
and $b = -1 - a = 2$

6. Find the points on the curve

$$y = 2x^3 + 3x^2 - 12x + 3$$

where the tangent line is horizontal.

$$\begin{aligned} f'(x) &= 6x^2 + 6x - 12 \\ &= 6(x^2 + x - 2) \\ &= 6(x+2)(x-1) \end{aligned}$$

Horizontal tangent lines occur when $f(x) = 0$:

$(1, f(1)) = (1, 4)$ $\quad (-2, f(-2)) = (-2, 23)$

• Implicit Differentiation

- The basic idea: If two expressions are equal, then so are their derivatives.
- Implicit differentiation is an application of the chain rule.
- Given an implicit equation involving x and y, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.
- Given an implicit equation involving x and y, find the equation of the tangent line at a point (x_0, y_0) on the curve.
- Implicit differentiation was used to find the derivative of $\ln(x)$ and $\log_a(x)$.
- Whenever the variable being differentiated differs from the variable that we are differentiating with respect to, a new derivative term is produced. For example, $\frac{d}{dz}(r^3) = 3r^2\frac{dr}{dz}$.

1. Find $\frac{dy}{dx}$, $\frac{dx}{dy}$ and $\frac{dx}{dt}$ for each equation:

(a) $xy + x^2y^2 = 6$

$$\frac{dy}{dx}: \underbrace{y + x\frac{dy}{dx}}_{\text{Product rule}} + \underbrace{2xy^2 + 2x^2y\frac{dy}{dx}}_{\text{Product rule}} = 0$$

Regroup:

$$y + 2xy^2 = -(x + 2x^2y)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = -\frac{y + 2xy^2}{x + 2x^2y}}$$

$$\frac{dx}{dy}: \boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = -\frac{x + 2x^2y}{y + 2xy^2}}$$

$$\frac{dx}{dt}: y\frac{dx}{dt} + x\frac{dy}{dt} + 2xy^2\frac{dx}{dt} + 2x^2y\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{x\frac{dy}{dt} + 2x^2y\frac{dy}{dt}}{y + 2xy^2}$$

(b) $e^{xy} = \sqrt[3]{xy^2}$

$$\frac{dy}{dx}: \underbrace{e^{xy}(y + x\frac{dy}{dx})}_{\text{Product rule}} = \underbrace{\frac{1}{3}(xy^2)^{-2/3}\left(x(2y)\frac{dy}{dx} + y^2\right)}_{\text{Chain rule then product rule}}$$

$$\frac{dy}{dx} = \frac{ye^{xy} - \frac{1}{3}(xy^2)^{-2/3}y^2}{-xe^{xy} + \frac{2xy}{3}(xy^2)^{-2/3}}$$

$$\boxed{\frac{dy}{dx} = \frac{3(xy^2)^{2/3}ye^{xy} - y^2}{-3(xy^2)^{2/3}xe^{xy} + 2xy}}$$

$$\frac{dx}{dy}: \boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{-xe^{xy} + \frac{2xy}{3}(xy^2)^{-2/3}}{ye^{xy} - \frac{1}{3}(xy^2)^{-2/3}y^2}}$$

$$\frac{dx}{dt}: \underbrace{e^{xy}(y\frac{dx}{dt} + x\frac{dy}{dt})}_{\text{Product rule}} = \underbrace{\frac{1}{3}(xy^2)^{-2/3}\left(x(2y)\frac{dy}{dx} + y^2\frac{dx}{dt}\right)}_{\text{Chain rule then product rule}}$$

$$\frac{dx}{dt} = \frac{-xe^{xy} + \frac{2xy}{3}(xy^2)^{-2/3}}{ye^{xy} - \frac{1}{3}(xy^2)^{-2/3}y^2}\left(\frac{dy}{dt}\right)$$